Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 342 (2008) 629–637

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Chord power integrals and radial mean bodies ☆

Xiong Ge [a,*], Cheung Wing-Sum [b]

[a] *Department of Mathematics, Shanghai University, Shanghai 200444, PR China*
[b] *Department of Mathematics, The University of Hong Kong, Hong Kong*

Received 28 September 2007

Available online 23 December 2007

Submitted by H.R. Parks

Abstract

In this paper, we obtain a formula relating the chord power integrals of a convex body K and the dual quermassintegrals of its radial pth mean body R_pK. With this, a relation among the chord power integrals of a convex body K under dilation transformations is found. As an interesting application, some geometric inequalities between the dual quermassintegrals of R_pK and the volume of K, which are equivalent to the isoperimetric-type inequalities of chord power integrals, are also established.

Keywords: Convex body; Dual quermassintegrals; Chord power integrals; Radial mean body

1. Introduction

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. Denote by $AG_{i,n}$ the affine Grassmann manifold of i-dimensional planes in $\mathbb{R}^n$. It is a homogeneous space under the action of the motion group $G(n)$ (see [6, p. 199]). Let $d\xi_k$ be the normalized invariant measure of $AG_{i,n}$ whose restriction to the Grassmann manifold $G_{i,n}$ is the invariant probability measure. Let K be a convex body in $\mathbb{R}^n$, and let ξ_1 be a random line intersecting K. Then $V_1(K \cap \xi_1)$ is the chord length of the intersection $K \cap \xi_1$, where V_1 is the 1-dimensional Lebesgue measure. The *chord power integrals* of K are defined by

$$I_p(K) = \frac{2\alpha_{n-1}}{n} \int\limits_{\xi_1 \in AG_{1,n}} V_1(K \cap \xi_1)^p \, d\xi_1, \quad 0 \leqslant p < \infty, \tag{1.0}$$

where α_{n-1} is the surface area of the unit sphere S^{n-1}. Note that the normalization says that

☆ Supported in part by the Youth Science Foundation of Shanghai (Grant No. 214511), the second author is supported in part by the Research Grants Council of the Hong Kong SAR, China (Project No. HKU7016/07P).

* Corresponding author.
E-mail addresses: xiongge@staff.shu.edu.cn (G. Xiong), wscheung@hku.hk (W.S. Cheung).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.12.033

$$\int\limits_{B_n\cap\xi_1\neq\emptyset} d\xi_1=\omega_{n-1},$$

where B_n is the unit ball in $\mathbb{R}^n$ and ω_{n-1} is the volume of the unit ball B_{n-1} in $\mathbb{R}^{n-1}$.

Chord power integrals are generalizations of the surface area $S(K)$ and the volume $V(K)$ of a convex body K. There are several interesting integral formulas for chord power integrals (see [5,6,14]):

$$\begin{aligned} I_0(K)&=\frac{\omega_{n-1}}{n}S(K),\\ I_1(K)&=\frac{\alpha_{n-1}}{n}V(K),\\ I_{n+1}(K)&=(n+1)V(K)^2. \end{aligned}\tag{1.1}$$

In 1998, Gardner and Zhang [3] introduced the notion of radial pth mean body. For any convex body K in $\mathbb{R}^n$ and any nonzero $p>-1$, the *radial pth mean body* of K is the new geometric object R_pK defined by the relation

$$\rho_{R_pK}(u)=\left(\frac{1}{V(K)}\int\limits_K \rho_K(x,u)^p\,dx\right)^{\frac{1}{p}},\quad \forall u\in S^{n-1},\tag{1.2}$$

where $\rho_K(x,u)=\max\{c\colon\ x+cu\in K\}$. Intuitively speaking, the radial function of the radial pth mean body R_pK of K is just the pth mean of the values of the radial function of K with respect to points inside K. Since $R_\infty K$ is the *difference body DK* of K, the process of forming R_pK can be regarded as a generalization of central or radial symmetrization. Moreover, since the shape of R_pK tends to that of the *polar projection body* Π^*K as p tends to -1, the new bodies form a spectrum connecting the *difference body* and *polar projection body* (see [3]).

The authors of paper [3] established a strong and sharp affine inequality relating the volume of R_pK to that of R_qK when $-1<p<q$. When $p=n$ and $q\to\infty$, this becomes the Rogers–Shephard inequality, while as $p\to-1$ and $q=n$, it becomes the famous Zhang projection inequality.

In this paper, we first establish an equation relating the chord power integrals of a convex body K and the dual quermassintegrals of its radial mean body R_pK by using integral geometric techniques (Theorem 1). Then by proving the invariant property of radial mean body under nonsingular linear transformations, we obtain a homogeneity property of chord power integrals of the convex body K under dilation transformations (Theorem 2) and prove that the radial pth mean body of a centered ellipsoid is still a centered ellipsoid (Theorem 3). In Theorem 4, we prove new geometric inequalities involving dual quermassintegrals $\widetilde{W}_{n-p}(R_pK)$ and the volume $V(K)$ of a convex body K, which are actually equivalent to the isoperimetric-type inequalities of chord power integrals.

For quick reference and to make the paper self-contained, we recall some basic facts of convex bodies in the next section.

2. Definitions and preliminaries

As usual, S^{n-1} denotes the unit sphere, B_n the unit ball and o the origin in the n-dimensional Euclidean space $\mathbb{R}^n$. The surface area of the unit sphere S^{n-1} and the volume of the unit ball B_n in $\mathbb{R}^n$ are denoted by α_{n-1} and ω_n, respectively. Note that $\alpha_{n-1}=n\omega_n$, and

$$\omega_n=\frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.$$

This suggests that we define

$$\omega_p=\frac{2\pi^{\frac{p}{2}}}{p\Gamma(\frac{p}{2})}\quad \text{for any } p>0.$$

By a direction in $\mathbb{R}^n$, we mean a unit vector, that is, an element of S^{n-1}. If u is a direction, we denote by $u^\perp$ the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to u. The line through o parallel to u is denoted by l_u.

Let K be a convex body in $\mathbb{R}^n$. For $u \in S^{n-1}$ and $y \in u^{\perp}$, let

$$X_u K(y) = V_1\big(K \cap (l_u + y)\big),$$

the function is called the *X-ray* of K in the direction u, where V_1 is the 1-dimensional Lebesgue measure. See, for example, [2, Chapter 1].

Let K be a star-shaped compact set with respect to a point z in $\mathbb{R}^n$. The *radial function* ρ_K is defined, for all $u \in S^{n-1}$ such that the line through z parallel to u intersects K, by

$$\rho_K(z,u) = \max\{c\colon\ z + cu \in K\}.$$

When z is the origin, we usually denote $\rho_K(o,u)$ by $\rho_K(u)$. By a *star body* we mean a star-shaped compact set K whose radial function $\rho_K(z,u)$ is continuous with respect to u. The *dual quermassintegrals* of a star body K are defined by

$$\widetilde{W}_{n-i}(K,z) = \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^i(u,z)\,du, \quad 0 \leqslant i \leqslant n, \tag{2.0}$$

or, more generally,

$$\widetilde{W}_{n-r}(K,z) = \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^r(z,u)\,du, \quad r \in \mathbb{R}. \tag{2.1}$$

The dual quermassintegrals were recently discovered by Lutwak (see [4]). They play the roles in the study of cross-sections of convex bodies as the mixed volumes do for the study of projections of convex bodies. One is referred to the books [2] and [7] for excellent illustrations. Dual mixed volumes are far from well understood. Their applications to the characterizations of intersection bodies and the solutions of the Busemann–Petty problems are very recent developments (see [13]).

Next, we give another integral formula for the chord power integrals which will be useful later.

The position of an oriented line ξ_1 in $\mathbb{R}^n$ can be characterized by specifying its direction u and the point r where ξ_1 intersects the orthogonal hyperplane of u through the origin. The "integral-geometric density" $d\xi_1$, for oriented lines is then given by

$$d\xi_1 = dr\,du,$$

where dr is the $(n-1)$-dimensional volume element in the orthogonal hyperplane $u^{\perp}$, and du is the element of surface area of the unit sphere S^{n-1}. Let $\sigma(r,u) = V_1(K \cap \xi_1)$, then we have

$$I_\lambda(K) = \frac{1}{n}\int\limits_{S^{n-1}}\ \int\limits_{K\cap u^{\perp}} \sigma(r,u)^{\lambda}\,dr\,du, \quad 0 \leqslant \lambda < \infty. \tag{2.2}$$

The *difference body* of the convex body K, denoted by DK, is the centrally symmetric convex body (centered at the origin) defined by

$$DK = K + (-K) = \{x - y\colon\ x \in K,\ y \in K\}.$$

It is well known that DK can be equivalently described as

$$DK = \big\{x\colon\ (x + K) \cap K \neq \emptyset\big\}.$$

The *covariogram* $g_K(x)$ of the convex body K in $\mathbb{R}^n$ is the function

$$g_K(x) = V\big(K \cap (K + x)\big), \quad x \in \mathbb{R}^n.$$

If K is a convex body that contains the origin in its interior, it is not difficult to verify that the *radial function* ρ_{DK} of the difference body DK is

$$\rho_{DK}(u) = \max_{y\in u^{\perp}} V_1\big(K \cap (l_u + y)\big), \quad u \in S^{n-1}.$$

Let

$$E_K(r,u) = \{ y \in u^{\perp} \colon X_u K(y) \geqslant r \},$$

the restricted chord projection [9,10] of K is the function

$$a_K(r,u) = V_{n-1}\big(E_K(r,u)\big), \quad r \geqslant 0,\ u \in S^{n-1}.$$

3. Main results

Theorem 1. *Let K be a convex body in $\mathbb{R}^n$. Then for $p > -1$,*

$$I_{p+1}(K) = (p+1)V(K)\widetilde{W}_{n-p}(R_p K).$$

Proof. Let $\xi_1^\star$ be an oriented line whose direction is $u \in S^{n-1}$. Let z be a point in the convex body K. Denote by t the distance of z to the boundary of K along the direction u. Suppose that $\xi_1^\star$ passes through z. Then $d\xi_1^\star\, dt = \alpha_{n-1}^{-1}\, dz\, du$ (see [6, p. 207]). Multiplying both sides by t^p and integrating over K, we have

$$\int\limits_{K\cap\xi_1^\star \neq \emptyset} \left(\int\limits_0^\sigma t^p\, dt \right) d\xi_1^\star = \frac{1}{\alpha_{n-1}} \int\limits_{z\in K} \left(\int\limits_{S^{n-1}} t^p\, du \right) dz.$$

So,

$$\frac{1}{p+1} \int\limits_{K\cap\xi_1^\star \neq \emptyset} \sigma^{p+1}\, d\xi_1^\star = \frac{1}{\alpha_{n-1}} \int\limits_{z\in K} \int\limits_{S^{n-1}} \rho_K^p(z,u)\, du\, dz.$$

From the definition of radial pth mean body (1.2) and Fubini's theorem, we have

$$\begin{aligned} \frac{\alpha_{n-1}}{n} \int\limits_{K\cap\xi_1^\star \neq \emptyset} \sigma^{p+1}\, d\xi_1^\star &= \frac{(p+1)V(K)}{n} \int\limits_{S^{n-1}} \rho_{R_p K}^p(u)\, du \\ &= (p+1)V(K)\widetilde{W}_{n-p}(R_p K), \end{aligned}$$

then it precisely gives

$$\begin{aligned} I_{p+1}(K) &= \frac{2\alpha_{n-1}}{n} \int\limits_{\xi_1 \in AG_{1,n}} V_1(K\cap\xi_1)^{p+1}\, d\xi_1 = \frac{\alpha_{n-1}}{n} \int\limits_{K\cap\xi_1^\star \neq \emptyset} \sigma^{p+1}\, d\xi_1^\star \\ &= (p+1)V(K)\widetilde{W}_{n-p}(R_p K), \quad p > -1. \end{aligned}$$

This completes the proof of the theorem. □

Remark. Here, we must point out that we use the technique of integral geometry to prove our result, where we benefit much from Zhang's Lemma 3.1 in [12]. In fact, we also can prove the above equation for $p > 0$ by the following lemma proved by the authors of [3] (see [3, Lemma 3.1]), which will also be used later.

Lemma 1. *(See [3].) Let K be a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$.*

(i) *For $p > -1$, we have*

$$\int\limits_K \rho_K(x,u)^p\, dx = \int\limits_0^{\rho_{DK}(u)} a_K(r,u) r^p\, dr.$$

(ii) *For $p > 0$, we have*

$$\int_K \rho_K(x,u)^p\,dx = p\int_0^{\rho_{DK(u)}} g_K(ru)r^{p-1}\,dr.$$

Second proof of Theorem 1 (for $p > 0$). From the formula (2.2) of chord power integrals, we have

$$I_{p+1}(K) = \frac{1}{n}\int_{S^{n-1}}\int_{K\cap u^\perp} \sigma(r,u)^{p+1}\,dr\,du, \quad -1 \leqslant p < \infty.$$

On the other hand, from the proof of Theorem 3 in [1] (also see [8]), we have

$$\int_0^{\rho_{DK}(u)} g_K(ru)r^{p-1}\,dr = \frac{1}{p(p+1)}\int_{K\cap u^\perp} \sigma(r,u)^{p+1}\,dr, \quad p > -1.$$

Hence, we have

$$I_{p+1}(K) = \frac{p(p+1)}{n}\int_{S^{n-1}}\int_0^{\rho_{DK}(u)} g_K(ru)r^{p-1}\,dr\,du, \quad p > -1.$$

From Lemma 1(ii) and Fubini's theorem, we have

$$\begin{aligned} I_{p+1}(K) &= \frac{p+1}{n}\int_{S^{n-1}}\int_K \rho_K(x,u)^p\,dx\,du = \frac{(p+1)V(K)}{n}\int_{S^{n-1}} \rho_{R_pK}(u)^p\,du \\ &= (p+1)V(K)\widetilde{W}_{n-p}(R_pK) \end{aligned}$$

for $p > 0$. This completes the proof of Theorem 1. □

Corollary 1. *Let K be a convex body in $\mathbb{R}^n$. Then $V(R_n(K)) = V(K)$.*

Proof. Letting $p = n$ in Theorem 1, we have

$$I_{n+1}(K) = (n+1)V(K)\widetilde{W}_0(R_nK).$$

Since

$$I_{n+1}(K) = (n+1)V(K)^2, \qquad \widetilde{W}_0(R_nK) = V(R_nK),$$

we have

$$V\big(R_n(K)\big) = V(K).$$

Hence the corollary. □

Corollary 2. *Let K and L be convex bodies in $\mathbb{R}^n$. If $R_pK = R_pL$ for all $p > -1$, then $I_{p+1}(K) = I_{p+1}(L)$ for all $p > -1$.*

Proof. From Theorem 1, for all $p > -1$ we have

$$I_{p+1}(K) = (p+1)V(K)\widetilde{W}_{n-p}(R_pK)$$

and

$$I_{p+1}(L) = (p+1)V(L)\widetilde{W}_{n-p}(R_pL).$$

Since

$$R_p K = R_p L \quad \text{for all } p > -1,$$

by Corollary 1 we have

$$V(K) = V(R_n K) = V(R_n L) = V(L).$$

Since

$$\widetilde{W}_{n-p}(R_p K) = \widetilde{W}_{n-p}(R_p L) \quad \text{for all } p > -1,$$

we have

$$I_{p+1}(K) = I_{p+1}(L), \quad p > -1.$$

This completes the proof of Corollary 2. □

From the Minkowski inequality for dual mixed volumes, the following inequalities are immediate.

Corollary 3. *Let K be a convex body in $\mathbb{R}^n$. Then*

(i) *for $0 < p < n$, $I_{p+1}(K) \leqslant (p+1)\omega_n^{\frac{n-p}{n}} V(K) V^{\frac{p}{n}}(R_p K)$, where the equality holds if and only if $R_p K$ is a ball*;
(ii) *for $p > n$, or $-1 < p < 0$, $I_{p+1}(K) \geqslant (p+1)\omega_n^{\frac{n-p}{n}} V(K) V^{\frac{p}{n}}(R_p K)$, where the equality holds if and only if $R_p K$ is a ball.*

Proof. From Theorem 1, for all $p > -1$ we have

$$I_{p+1}(K) = (p+1) V(K) \widetilde{W}_{n-p}(R_p K).$$

According to the dual Minkowski inequality, we have

$$\begin{aligned} I_{p+1}(K) &\leqslant (p+1) V(K) V^{\frac{p}{n}}(R_p K) V^{\frac{n-p}{n}}(B_n) \\ &= (p+1) V(K) \omega_n^{\frac{n-p}{n}} V^{\frac{p}{n}}(R_p K), \quad 0 < p < n, \end{aligned}$$

and

$$\begin{aligned} I_{p+1}(K) &\geqslant (p+1) V(K) V^{\frac{p}{n}}(R_p K) V^{\frac{n-p}{n}}(B_n) \\ &= (p+1) V(K) \omega_n^{\frac{n-p}{n}} V^{\frac{p}{n}}(R_p K), \quad -1 < p < 0,\ p > n. \end{aligned}$$

From the equality condition of dual Minkowski inequality, we can conclude that both equalities hold in the above inequalities if and only if $R_p K$ is a centered n-dimensional ball. This completes the proof of Corollary 3. □

To characterize the property of chord power integrals under dilation of the convex body K, we first give a lemma about the invariant property for radial pth mean body under linear transformation.

Lemma 2. *Let K be a convex body in $\mathbb{R}^n$ and $GL(n)$ the nonsingular linear transformation group. Then for $\varphi \in GL(n)$ and $p > -1$, we have $R_p(\varphi K) = \varphi(R_p(K))$.*

Proof. For $\varphi \in GL(n)$, using the formula

$$\rho_{\varphi K}(x, u) = \rho_K\left(\varphi^{-1} x, \varphi^{-1} u\right)$$

of radial functions, we have

$$\begin{aligned}
\rho^p_{R_p(\varphi K)}(u) &= \frac{1}{V(\varphi K)} \int_{\varphi K} \rho_{\varphi K}(x,u)^p \, dx \\
&= \frac{1}{|\det\varphi| V(K)} \int_{\varphi K} \rho_K\big(\varphi^{-1}x, \varphi^{-1}u\big)^p \, dx \\
&= \frac{1}{|\det\varphi| V(K)} \int_{K} \rho_K\big(y, \varphi^{-1}u\big)^p |\det\varphi| \, dy \\
&= \rho^p_{R_p K}\big(\varphi^{-1}u\big) \\
&= \rho^p_{\varphi(R_p K)}(u),
\end{aligned}$$

this completes the proof of Lemma 2. □

Remark. In [3, Theorem 2.3], the authors proved the same relation for volume-preserving linear transformations.

Theorem 2. *Let K be a convex body in $\mathbb{R}^n$. Then for all $p > -1$, we have $I_{p+1}(cK) = c^{n+p} I_{p+1}(K)$, $\forall c \geqslant 0$.*

Proof. From Theorem 1, for all $p > -1$ we have

$$I_{p+1}(K) = (p+1)V(K)\widetilde{W}_{n-p}(R_p K).$$

From Lemma 2 and the property of dual mixed volume, we have

$$\begin{aligned}
I_{p+1}(cK) &= (p+1)V(cK)\widetilde{W}_{n-p}\big(R_p(cK)\big) \\
&= (p+1)c^n V(K)\widetilde{W}_{n-p}(cR_p K) \\
&= (p+1)c^n V(K)c^p \widetilde{W}_{n-p}(R_p K) \\
&= c^{n+p}(p+1)V(K)\widetilde{W}_{n-p}(R_p K) \\
&= c^{n+p} I_{p+1}(K)
\end{aligned}$$

for all $c \geqslant 0$. This completes the proof of Theorem 2. □

From Lemma 2, we can prove that the radial pth mean body of a centered ellipsoid is still a centered ellipsoid. Then with Theorem 1, we can compute the chord power integrals of the unit ball easily.

Theorem 3. *Let K be a centered ellipsoid in $\mathbb{R}^n$. Then $R_p K$ is still a centered ellipsoid for all $p > -1$.*

Proof. We first consider the radial pth mean body of the unit ball B_n. Obviously, we have

$$a_{B_n}(r,u) = V_{n-1}\left(\sqrt{1 - \left(\frac{r}{2}\right)^2} B_{n-1}\right), \qquad \rho_{DB_n}(u) = 2$$

for all $u \in S^{n-1}$. From Lemma 1(i), we have

$$\begin{aligned}
\rho^p_{R_p B_n}(u) &= \frac{1}{V(B_n)} \int_{B_n} \rho_{B_n}(x,u)^p \, dx \\
&= \frac{1}{V(B_n)} \int_0^{\rho_{DB_n}(u)} a_{B_n}(r,u) r^p \, dr \\
&= \frac{\omega_{n-1}}{\omega_n} \int_0^2 \left(1 - \frac{r^2}{4}\right)^{\frac{n-1}{2}} r^p \, dr
\end{aligned}$$

$$= \frac{2^p \omega_{n-1}}{\omega_n} B\left(\frac{n+1}{2}, \frac{p+1}{2}\right)$$

for all $p > -1$. Let

$$c_{n,p} = \left\{\frac{2^p \omega_{n-1}}{\omega_n} B\left(\frac{n+1}{2}, \frac{p+1}{2}\right)\right\}^{\frac{1}{p}},$$

we have

$$\rho_{R_p B_n}(u) = c_{n,p},$$

which is a constant depending only on the numbers p and n. Hence the radial pth mean body of unit ball B_n is still a ball centered at the origin with radius $c_{n,p}$.

When K is a centered ellipsoid, there exists $\varphi \in GL(n)$ such that $K = \varphi B_n$. From Lemma 2 and the above argument, we have

$$R_p K = R_p(\varphi B_n) = \varphi(R_p B_n) = \varphi(c_{n,p} B_n) = c_{n,p}\varphi(B_n) = c_{n,p} K,$$

which is still a centered ellipsoid. This completes the proof of Theorem 3. □

Corollary 4. *Let B_n be the unit ball in $\mathbb{R}^n$. Then for $p > -1$, we have $I_{p+1}(B_n) = \frac{2^{p+1}\omega_n \omega_{p+n}}{\omega_{p+1}}$.*

Proof. From Theorems 1 and 3, we have

$$\begin{aligned} I_{p+1}(B_n) &= (p+1)V(B_n)V^{\frac{p}{n}}(R_p B_n)V^{\frac{n-p}{n}}(B_n) \\ &= (p+1)\omega_n^{\frac{2n-p}{n}} V^{\frac{p}{n}}(c_{n,p} B_n) \\ &= (p+1)c_{n,p}^p \omega_n^2 \\ &= \frac{2^{p+1}\omega_n \omega_{p+n}}{\omega_{p+1}}. \qquad \square \end{aligned}$$

As an application of Theorems 1 and 2, we can derive new inequalities involving dual quermassintegrals $\widetilde{W}_{n-p}(R_p K)$ and the volume $V(K)$ of a convex body K by using the isoperimetric inequalities of chord power integrals. For this aim, we narrate the known results in the following

Proposition 1. *If B is the ball with the same volume as the convex body K in $\mathbb{R}^n$, then*

$$I_\lambda(K) \geqslant I_\lambda(B), \quad 0 \leqslant \lambda < 1, \tag{3.1}$$

$$I_1(K) = I_1(B), \quad \lambda = 1, \tag{3.2}$$

$$I_\lambda(K) \leqslant I_\lambda(B), \quad 1 < \lambda < n+1, \tag{3.3}$$

$$I_{n+1}(K) = I_{n+1}(B), \quad \lambda = n+1, \tag{3.4}$$

$$I_\lambda(K) \geqslant I_\lambda(B), \quad n+1 < \lambda < \infty. \tag{3.5}$$

The equality in either (3.3) *or* (3.5) *holds if and only if K is a ball. The equality in* (3.1) *holds if K is a ball.*

Eqs. (3.2) and (3.4) are the integral formulas of Crofton and Santaló. When λ are positive integers, these inequalities were proved by Blaschke ($n = 2$), Wu ($n = 3$), and Ren ($n > 3$). The general case above was proved in [11] by Zhang.

Combined with Theorems 1, 2, Corollary 4 and Proposition 1, we can obtain the following inequalities directly, including the condition for equality.

Theorem 4. *If K is a convex body in $\mathbb{R}^n$, then*

$$\widetilde{W}_{n-p}(R_pK) \geqslant \frac{2^{p+1}V(K)^{\frac{p}{n}}\omega_{n+p}}{(p+1)\omega_n^{\frac{p}{n}}\omega_{p+1}}, \quad -1 < p < 0, \tag{3.6}$$

$$\widetilde{W}_{n-p}(R_pK) \leqslant \frac{2^{p+1}V(K)^{\frac{p}{n}}\omega_{n+p}}{(p+1)\omega_n^{\frac{p}{n}}\omega_{p+1}}, \quad 0 < p < n, \tag{3.7}$$

$$\widetilde{W}_{n-p}(R_pK) \geqslant \frac{2^{p+1}V(K)^{\frac{p}{n}}\omega_{n+p}}{(p+1)\omega_n^{\frac{p}{n}}\omega_{p+1}}, \quad n < p < \infty. \tag{3.8}$$

The equality in either (3.7) *or* (3.8) *holds if and only if K is a ball. The equality in* (3.6) *holds if K is a ball.*

When $p \to -1^+$, inequality (3.6) becomes the classical Euclidean isoperimetric inequality. In fact, since

$$\big((p+1)V(K)\big)^{\frac{1}{p}} R_pK \to \Pi^*K, \quad p \to -1^+,$$

where Π^*K is the polar projection body of convex body K (see [3, Theorem 2.2]), and

$$W_{n+1}\big(\Pi^*K\big) = \frac{\omega_{n-1}}{n} S(K),$$

so from (3.6), we have

$$\left(\frac{S(K)}{\alpha_{n-1}}\right)^n \geqslant \left(\frac{V(K)}{\omega_n}\right)^{n-1}.$$

Remark. By the formula in Theorem 1, we note that the inequalities obtained in Theorem 4 are actually equivalent to the isoperimetric-type inequalities of chord power integrals in Proposition 1, and thus can also be viewed as isoperimetric-type inequalities.

Acknowledgments

The first author is most grateful to Professor Delin Ren for his always encouragement and help. We would also like to thank the referee(s) for many helpful comments.

References

[1] G.D. Chakerian, Inequalities for the difference body of a convex body, Proc. Amer. Math. Soc. 18 (1967) 879–884.
[2] R.J. Gardner, Geometric Tomography, Cambridge Univ. Press, Cambridge, 1995.
[3] R.J. Gardner, G. Zhang, Affine inequalities and radial mean bodies, Amer. J. Math. 120 (3) (1998) 505–528.
[4] E. Lutwak, Dual mixed volumes, Pacific J. Math. 58 (1975) 531–538.
[5] Ren Delin, Topics in Integral Geometry, World Scientific, River Edge, NJ, 1994.
[6] L.A. Santaló, Integral Geometry and Geometric Probability, Addison–Wesley, 1976.
[7] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge Univ. Press, Cambridge, 1993.
[8] Xiong Ge, Inequalities for chord power integrals, J. Korean Math. Soc. (2008), in press.
[9] G. Zhang, Restricted chord projection and affine inequalities, Geom. Dedicata 39 (1991) 213–222.
[10] G. Zhang, Integral geometric inequalities, Acta Math. Sinica 34 (1991) 72–90.
[11] G. Zhang, Geometric inequalities and inclusion measures of convex bodies, Mathematika 41 (1994) 95–116.
[12] G. Zhang, Dual kinematics formulas, Trans. Amer. Math. Soc. 351 (1999) 985–995.
[13] G. Zhang, A positive solution to the Busemann–Petty problem in R^4, Ann. of Math. 149 (1999) 535–543.
[14] G. Zhang, J. Zhou, Containment measures in integral geometry, in: E.L. Grinberg, et al. (Eds.), Integral Geometry and Convexity, World Scientific, 2006, pp. 153–168.